MILNOR'S K-THEORY AND FUNCTION FIELDS OF HYPERSURFACES IN POSITIVE CHARACTERISTIC

STEPHEN SCULLY

Abstract. Let X be an integral affine hypersurface over a field F of characteristic $p > 0$. Using a recent result of A. Dolphin and D. Hoffmann, we give an explicit description of the kernel of the restriction homomorphism $K_*^M(F)/pK_*^M(F) \to K_*^M(F(X))/pK_*^M(F(X))$, where $F(X)$ is the function field of X.

1. Introduction

Let F be a field, p a prime integer, and write $k_n(F)$ for the n^{th} Milnor K-group $K_n^M(F)/pK_n^M(F)$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients (cf. §2.1). If X is an (integral) algebraic variety over F, then for any $n \geq 0$, there is a restriction homomorphism

$$r_{F(X)/F}\colon k_n(F) \to k_n(F(X)),$$

where $F(X)$ denotes the function field of X. This map is of natural interest as a component of K. Kato's complex

$$k_n(F) \xrightarrow{r_{F(X)/F}} k_n(F(X)) \to \oplus_{x\in X^{(1)}} k_{n-1}(F(x)) \to \oplus_{x\in X^{(2)}} k_{n-2}(F(x)) \to \ldots$$

which carries important information about the variety X. On the other hand, the study of the maps $r_{F(X)/F}$ is further motivated by the appearance of the Milnor K-groups in several fundamental problems in algebra. For example:

(1) If $p \neq \mathrm{char}(F)$, then for any n, there is a natural isomorphism between $k_n(F)$ and the Galois cohomology group $H^n(F, \mu_p^{\otimes n})$. This is the content of the Bloch-Kato conjecture, recently proved by M. Rost and V. Voevodsky (cf. [Voe11]).

(2) As a special case of (1), if $p \neq \mathrm{char}(F)$ and F contains a primitive p^{th} root of unity, then there is a natural isomorphism $k_n(F) \simeq {}_p\mathrm{Br}(F)$, where ${}_p\mathrm{Br}(F)$ denotes the p-torsion subgroup of the Brauer group of F (this result was proved earlier by A. Merkurjev and A. Suslin; cf. [MS82]).

(3) If $p = 2$, then for all n, $k_n(F)$ is naturally isomorphic to the degree n component of the graded Witt ring $\mathrm{gr}_{I^\bullet}(W(F))$ of symmetric bilinear forms over F by results of D. Orlov, A. Vishik, V. Voevodsky (in the case $\mathrm{char}(F) \neq 2$; cf. [OVV07]), and K. Kato (in the case $\mathrm{char}(F) = 2$; cf. [Kat82b]).

In this context, it is of particular interest to determine the kernels $k_n(F(X)/F)$ of the maps $r_{F(X)/F}$, at least for certain classes of varieties X. If X is unirational, then $k_n(F(X)/F) = 0$ for all $n \geq 0$ by a result of J. Milnor (cf. [Mil70], Theorem 2.3). Beyond this observation, little is known.

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In the case where $p \neq \mathrm{char}(F)$, the problem is in general very difficult. For example, determining the groups $k_n(F(X)/F)$ in the special case where $p = 2 \neq \mathrm{char}(F)$ and X is a non-rational *quadric* is already one of the major outstanding problems in the algebraic theory of quadratic forms. In this case, it is generally conjectured that the groups $k_n(F(X)/F)$ are generated by *pure symbols* (cf. §2.1) for all n. If one could prove this claim, then much insight could be obtained into several fundamental problems; for instance, a complete motivic characterisation of Pfister neighbours would follow (see [Vis] for further discussion). The conjecture has been verified in [OVV07] for the special case where X is a *norm quadric*. This result relies however on V. Voevodsky's (highly nontrivial) proof of the Milnor conjecture, and for quadrics of other type, little is known.

In the case where $p = \mathrm{char}(F)$, the picture is rather simpler, due to a result of S. Bloch, O. Gabber and K. Kato, which states that the *differential symbol* *dlog*: $k_n(F) \to \Omega^n_F$ induces a natural isomorphism from $k_n(F)$ onto the kernel $\nu(n)_F$ of the Artin-Schreier operator $\wp\colon \Omega^n_F \to \Omega^n_F/B^n_F$ on the space Ω^n_F of absolute differential forms over F (cf. §2, in particular Theorem 2.2). Since we can make use of the linear structure on Ω^n_F, it is typically easier to prove statements on the level of the groups $\nu(n)_F$ than the K-groups $k_n(F)$. For example, if our variety X is geometrically reduced (that is, the function field $F(X)$ is separably generated over F), then it is a well known fact that the natural restriction homomorphisms $r_{F(X)/F}\colon \Omega^n_F \to \Omega^n_{F(X)}$ are injective (cf. [Mat89], Theorem 26.6). In view of the Bloch-Gabber-Kato theorem (Theorem 2.2), we therefore have:

Proposition 1.1. *Let $p = \mathrm{char}(F)$, and let X be a geometrically reduced, integral variety over F. Then $k_n(F(X)/F) = 0$ for all n.*

Thus, if $p = \mathrm{char}(F)$, our problem is only interesting in the case where X is geometrically nonreduced, or equivalently, in the case where $F(X)$ is inseparable over F. In a recent article [DH], A. Dolphin and D. Hoffmann have computed the kernels $\Omega^n(F(X)/F)$ of the natural restriction maps $r_{F(X)/F}\colon \Omega^n_F \to \Omega^n_{F(X)}$ in the case where $p = \mathrm{char}(F)$ and X is any geometrically nonreduced, integral affine hypersurface over F. To formulate the result, we need the following construction (cf. [DH], Theorem 10.3):

Let $X \subset \mathbb{A}^k_F$ be a geometrically nonreduced affine hypersurface over F, defined by the vanishing of some $f \in \mathcal{O}(\mathbb{A}^k_F)$. Choose some coordinates on $\mathbb{A}^k_F$ so that f is identified with a polynomial in k variables $T_1, ..., T_k$ over F. With respect to this identification, let $\widehat{\mathcal{C}}$ be the set of all ratios of the nonzero coefficients of f. We define $N_F(X) = F^p(\widehat{\mathcal{C}})$. By definition, $N_F(X)$ is finite dimensional over its subfield F^p, and we have $[N_F(X) : F^p] = p^n$ for some $n \geq 0$. If X is additionally integral, then $[N_F(X) : F^p] > 1$, because otherwise f would be reducible.

Remark 1.2. The field $N_F(X)$ can be constructed in a more intrinsic fashion as follows: The element $f \in \mathcal{O}(\mathbb{A}^k_F)$ may be regarded as a function $f\colon \mathbb{A}^n_F(F) \to F$. Let $D(f) \subset F$ be the image of this map. Then we have

$$N_F(X) = F^p(\frac{a}{b} \mid a, b \in D(f) \cap F^*).$$

In other words, $N_F(X)$ is the subfield of F generated over F^p by all ratios of nonzero values of f. This justifies the use of the notation $N_F(X)$. Note however that this construction still depends on the choice of embedding of X as a hypersurface in some affine space.

Theorem 1.3 (Dolphin-Hoffmann, [DH], Theorem 10.3)**.** *Let* $\mathrm{char}(F) = p > 0$, *and let* X *be a geometrically nonreduced, integral affine hypersurface over* F. *Let* $n \geq 1$ *be such that* $[N_F(X) : F^p] = p^n$, *and let* $a_1, ..., a_n \in F^*$ *be such that* $N_F(X) = F^p(a_1, ..., a_n)$. *Then*

$$\Omega^m(F(X)/F) = \begin{cases} \Omega_F^{m-n} \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$

In the present article, we use this theorem to prove the following result:

Main Theorem. *Let* $\mathrm{char}(F) = p > 0$, *and let* X *be a geometrically nonreduced, integral affine hypersurface over* F. *Let* $n \geq 1$ *be such that* $[N_F(X) : F^p] = p^n$. *Then*

(1) *If* $m \geq n$, *then* $k_m(F(X)/F)$ *is the subgroup of* $k_m(F)$ *generated by pure symbols* $\{b_1, ..., b_m\}$ *satisfying* $N_F(X) \subset F^p(b_1, ..., b_m)$.
(2) *If* $m < n$, *then* $k_m(F(X)/F) = 0$.

In particular, the kernels $k_m(F(X)/F)$ are generated by pure symbols (compare with the aforementioned conjecture on quadrics in the case where $p = 2 \neq \mathrm{char}(F)$). The theorem is an immediate consequence of Theorem 1.3, the Bloch-Gabber-Kato theorem (Theorem 2.2), and the following result:

Theorem 1.4. *Let* F *be a field of characteristic* $p > 0$, *let* $a_1, ..., a_n \in F$ *be such that* $[F^p(a_1, ..., a_n) : F^p] = p^n$ *for some* $n \geq 1$, *and let* $m \geq n$. *Then* $\nu(m)_F \cap (\Omega_F^{m-n} \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n})$ *is equal to the subgroup of* $\nu(m)_F$ *generated by those elements* $dlog(\{b_1, ..., b_m\})$ *which satisfy* $F^p(a_1, ..., a_n) \subset F^p(b_1, ..., b_m)$.

This result is a refined version of the theorem of K. Kato which states that differential symbol *dlog* maps $k_m(F)$ surjectively onto the group $\nu(m)_F$. The proof of Theorem 1.4, which uses the methods developed by K. Kato in [Kat82b], can be found in §5.

2. Basic notions

2.1. **Milnor's K-theory.** All statements which appear in this section can be found in ([GS06], Chapter 7) Let F be a field. The *Milnor ring* of F (introduced in [Mil70]) is the quotient

$$K_*^M(F) = T_*(F^*)/(a \otimes (1-a) \mid a \in F^* \setminus \{1\})$$

of the tensor ring $T_*(F^*)$ modulo specified quadratic relations. The ring $K_*^M(F)$ is graded commutative with respect to the natural grading it inherits from the tensor ring $T_*(F^*)$. Its n^{th} graded component, $K_n^M(F)$, is called the n^{th} Milnor K-group of F. We have $K_n^M(F) = 0$ for $n < 0$, $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^*$. The image a pure tensor $a_1 \otimes ... \otimes a_n \in T_n(F^*)$ under the canonical surjection $T_n(F^*) \to K_n^M(F)$ is

denoted by $\{a_1, ..., a_n\}$. Elements of this type are called *pure symbols*. By definition, the Milnor K-groups are additively generated by pure symbols.

An extension of fields L/F induces a *restriction homomorphism*

$$r_{L/F}\colon K_*^M(F) \to K_*^M(L)$$

of graded rings in the obvious way. In particular, $K_*^M(L)$ has the natural structure of a left and right graded $K_*^M(F)$-module. The restriction homomorphisms are clearly functorial with respect to field extensions.

A *finite* extension of fields L/F induces a *transfer homomorphism*

$$N_{L/F}\colon K_*^M(L) \to K_*^M(F)$$

of graded abelian groups. These maps are functorial with respect to field extensions, and the composition $N_{L/F} \circ r_{L/F}$ coincides with multiplication by the degree $[L : F]$.

2.2. **Differential forms.** All statements which appear in this section can be found in ([Mat89], Chapter 9). Let $A \subset B$ be an extension of commutative rings. Recall that there is a unique pair $(\Omega^1_{B|A}, d)$ consisting of a B-module $\Omega^1_{B|A}$ and an A-linear derivation $d\colon B \to \Omega^1_{B|A}$ such that for any B-module M, the map $\phi \to \phi \circ d$ induces an isomorphism

$$\mathrm{Hom}_B(\Omega^1_{B|A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M)$$

of A-modules (where $\mathrm{Der}_A(B, M)$ is the A-module of all A-linear derivations $B \to M$). The B-module $\Omega^1_{B|A}$ is called the *module of differential forms of B relative to A*. Explicitly, the pair $(\Omega^1_{B|A}, d)$ can be constructed as follows: We take $\Omega^1_{B|A}$ to be the quotient of the free B-module generated by symbols db, $b \in B$, by the submodule generated by elements of the form

(1) da $(a \in A)$,
(2) $d(b_1 + b_2) - db_1 - db_2$ $(b_1, b_2 \in B)$, and
(3) $d(b_1b_2) - b_1db_2 - b_2db_1 (b_1, b_2 \in B)$.

The derivation $d\colon B \to \Omega^1_{B|A}$ is then given by the canonical map $b \mapsto db$.

For any $i > 0$, we define the B-module $\Omega^i_{B|A}$ as the i-fold exterior product $\bigwedge^i \Omega^1_{B|A}$. We also put $\Omega^0_{B|A} = B$ by convention. For all $i \geq 1$, there are A-module homomorphisms $d\colon \Omega^i_{B|A} \to \Omega^{i+1}_{B|A}$ extending the derivation $d\colon B = \Omega^0_{B|A} \to \Omega^1_{B|A}$, and satisfying

$$d(\omega \wedge \omega') = d(\omega) \wedge \omega' + (-1)^j \omega \wedge d(\omega')$$

for all $\omega \in \Omega^j_{B|A}$ and $\omega' \in \Omega^{i-j}_{B|A}$. If there exists a collection $\{b_j \mid j \in J\}$ of elements of B such that the elements $(db_j)_{j \in J}$ freely generate $\Omega^1_{B|A}$ as a B-module (for example, if B is a field), then d is simply defined by the formula

$$d(bdb_{j_1} \wedge ... \wedge db_{j_i}) = db \wedge db_{j_1} \wedge ... \wedge db_{j_i}. \tag{2.1}$$

The collection

$$(\Omega^\bullet_{B|A}, d) = B \xrightarrow{d} \Omega^1_{B|A} \xrightarrow{d} \Omega^2_{B|A} \xrightarrow{d} \Omega^3_{B|A} \xrightarrow{d} ...$$

of A-modules and A-module homomorphisms is a complex, called the *de Rham complex*. For all $i \geq 1$, we write $B^i_{B|A}$ for the image of the map $d\colon \Omega^{i-1}_{B|A} \to \Omega^i_{B|A}$. In

the special case where $A=\mathbb{Z}$, we will simply write Ω^i_B (resp. B^i_B) in place of $\Omega^i_{B|A}$ (resp. $B^i_{B|A}$).

If $A\subset B\subset C$ is a tower of extensions, then there is a canonical *restriction homomorphism*

$$r_{C|A/B|A}\colon \Omega^i_{B|A}\to\Omega^i_{C|A}$$

of B-modules for all $i\geq 0$. These maps commute with the differentials d, so we have induced homomorphisms $r_{C|A/B|A}\colon \Omega^i_{B|A}/B^i_{B|A}\to\Omega^i_{C|A}/B^i_{C|A}$. If $A=\mathbb{Z}$, we will simply write $r_{C/B}$ in place of $r_{C|A/B/A}$. The restriction homomorphisms are clearly functorial with respect to extensions of commutative A-algebras.

Let $k\subset F$ be an extension with k a noetherian commutative ring and F a field. If L/F is a *finite separable* extension of fields, then the canonical homomorphism $L\otimes_F\Omega^i_{F|k}\to\Omega^i_{L|k}$ is an isomorphism of L-vector spaces for all $i\geq 0$. This allows us to define a *transfer homomorphism*

$$Tr_{L|k/F|k}\colon \Omega^i_{L|k}\to\Omega^i_{F|k}$$

of abelian groups for all $i\geq 0$ as the composition

$$\Omega^i_{L|k}\xrightarrow{\sim} L\otimes_F\Omega^i_{F|k}\xrightarrow{Tr\otimes id}\Omega^i_{F|k},$$

where $Tr\colon L\to F$ denotes the classical trace homomorphism. These maps commute with the differentials d, so we have induced homomorphisms $Tr_{L|k/F|k}\colon \Omega^i_{L|k}/B^i_{L|k}\to \Omega^i_{F|k}/B^i_{F|k}$. If $k=\mathbb{Z}$, we will simply write $Tr_{L/F}$. These maps are functorial with respect to field extensions over k, and the composition $Tr_{L|k/F|k}\circ r_{L|k/F|k}$ coincides with multiplication by the degree $[L:F]$.

2.3. **Differential forms in positive characteristic.** All statements which appear in this section can be found in ([GS06], Chapter 9). Let $k\subset F$ be an extension of fields of characteristic $p>0$ satisfying $F^p\subset k$. An element of $\Omega^i_{F|k}$ having the form $\frac{db_1}{db_1}\wedge\ldots\wedge\frac{db_i}{b_i}$ for some $b_i\in F$ will be called a *logarithmic pure symbol*. Note that such elements belong to $B^i_{F|k}$ by (2.1). A finite subset $\{b_1,\ldots,b_i\}\subset F$ is called *p-independent over k* if $[k(b_1,\ldots,b_i):k]=p^i$. A subset $\mathcal{B}=\{b_j\mid j\in J\}$ of elements of F is called a *p-basis of F over k* if $F=k(\mathcal{B})$, and every finite subset $\{b_{j_1},\ldots,b_{j_i}\}\subset\mathcal{B}$ is p-independent over k. This condition holds if and only if the logarithmic elements $(\frac{db_j}{b_j})_{j\in J}$, form a basis of the F-vector space $\Omega^1_{F|k}$. Since $\Omega^1_{F|k}$ certainly has a basis consisting of elements of the form $\frac{db}{b}$, p-bases exist for any extension $k\subset F$.

Now assume that $k\subset F$ is finite. Let r be such that $[F:k]=p^r$ (the degree is always a power of p, since we assume that $F^p\subset k$), and choose a p-basis $\{b_1,\ldots,b_r\}$ of F over k. For all $i\in[1,r]$, let S_i be the set of all strictly increasing functions $[1,i]\to[1,r]$. For every $s\in S_i$, we define logarithmic pure symbols

$$\omega_s=\frac{db_{s(1)}}{b_{s(1)}}\wedge\ldots\wedge\frac{db_{s(i)}}{b_{s(i)}}\in\Omega^i_{F|k}.$$

Since $\{b_1,\ldots,b_r\}$ is a p-basis of F over k, the set $\{\omega_s\mid s\in S_i\}$ is a basis of the F-vector space $\Omega^i_{F|k}$. Given a multiindex $\alpha=(\alpha_1,\ldots,\alpha_r)\in[0,p-1]^r$, we write b^α for the element $b_1^{\alpha_1}\ldots b_r^{\alpha_r}\in F$. The set $\{b^\alpha\omega_s\mid s\in S_i,\alpha\in[0,p-1]^r\}$ is a basis for $\Omega^i_{F|k}$ viewed as a k-vector space. Let $\alpha\in[0,p-1]^r$ be a multiindex. For each $i\in[1,r]$,

we write $\Omega^i_{F|k}(\alpha)$ for the k-vector subspace of $\Omega^i_{F|k}$ generated by the elements $b^\alpha\omega_s$, $s \in S_i$. We also set $\Omega^0_{F|k}(\alpha)$ to be the 1-dimensional k-vector subspace of $\Omega^0_{F|k} = F$ generated by the element b^α. For each $i \geq 0$, the k-linear differential $d\colon \Omega^i_{F|k} \to \Omega^{i+1}_{F|k}$ maps $\Omega^i_{F|k}(\alpha)$ to $\Omega^{i+1}_{F|k}(\alpha)$, and so we get a complex $(\Omega^\bullet_{F|k}(\alpha), d)$ of k-vector spaces. This gives a direct sum decomposition of (finite) complexes

$$(\Omega^\bullet_{F|k}, d) \simeq \oplus_\alpha (\Omega^\bullet_{F|k}(\alpha), d) \tag{2.2}$$

where α runs over all multiindices in $[0, p-1]^r$. By formula (2.1), the differentials of the complex $\Omega^\bullet_{F|k}(0)$ (here 0 denotes the multiindex $(0, ..., 0)$) are trivial. Moreover, we have

Proposition 2.1. *In the above notation, the complexes $(\Omega^\bullet_{F|k}(\alpha), d)$ are acyclic for all $\alpha \neq 0$. In particular, if $[F : k] = p^r$, then $\Omega^r_{F|k}/B^r_{F|k}$ is a 1-dimensional k-vector space generated by the class of $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_r}{b_r}$.*

Proof. For the first statement, see [GS06], Proposition 9.4.6. The second statement follows, because $\Omega^{r+1}_{F|k} = 0$ and the differentials of the complex $\Omega^\bullet_{F|k}(0)$ are trivial. □

2.4. **The differential symbol.** Let $k \subset F$ be an extension of fields of characteristic $p > 0$ satisfying $F^p \subset k$. For all $i \geq 0$, there is a homomorphism

$$\wp\colon \Omega^i_{F|k} \to \Omega^i_{F|k}/B^i_{F|k} \tag{2.3}$$

of abelian groups (called the *Artin-Schreier operator*), which acts on elements of the form $b\frac{db_1}{b_1} \wedge ... \wedge \frac{db_i}{b_i}$, $b, b_j \in F$, by the formula

$$\wp(b\frac{db_1}{b_1} \wedge ... \wedge \frac{db_i}{b_i}) = (b^p - b)\frac{db_1}{b_1} \wedge ... \wedge \frac{db_i}{b_i} \pmod{B^i_{F|k}}$$

In other words, $\wp = \gamma - \pi$, where $\gamma\colon \Omega^i_{F|k} \to \Omega^i_{F|k}/B^i_{F|k}$ is the *inverse Cartier operator*, and $\pi\colon \Omega^i_{F|k} \to \Omega^i_{F|k}/B_{F|k}$ is the canonical projection (cf. [GS06], §9.4).

The kernel of $\wp\colon \Omega^i_{F|k} \to \Omega^i_{F|k}/B^i_{F|k}$ is denoted by $\nu(i)_{F|k}$. If $F \subset L$ is an extension of fields over k, then the diagram

$$\begin{array}{ccc} \Omega^i_{F|k} & \xrightarrow{\wp} & \Omega^i_{F|k}/B_{F|k} \\ {\scriptstyle r_{L|k/F|k}}\downarrow & & \downarrow{\scriptstyle r_{L|k/F|k}} \\ \Omega^i_{L|k} & \xrightarrow{\wp} & \Omega^i_{L|k}/B_{L|k} \end{array}$$

is commutative, so we have induced homomorphisms $r_{L|k/F|k}\colon \nu(i)_{F|k} \to \nu(i)_{L|k}$. If $F \subset L$ is a *finite separable* extension of fields over k, then the diagram

$$\begin{array}{ccc} \Omega^i_{L|k} & \xrightarrow{\wp} & \Omega^i_{L|k}/B_{L|k} \\ {\scriptstyle Tr_{L|k/F|k}}\downarrow & & \downarrow{\scriptstyle Tr_{L|k/F|k}} \\ \Omega^i_{F|k} & \xrightarrow{\wp} & \Omega^i_{F|k}/B_{F|k} \end{array}$$

is commutative, so we also have induced homomorphisms $Tr_{L|k/F|k}\colon \nu(i)_{L|k} \to \nu(i)_{F|k}$.

Finally, note that for any field F of characteristic $p > 0$, the F-vector space Ω^i_F coincides with $\Omega^i_{F|F^p}$, and so the maps $\wp\colon \Omega^i_F \to \Omega^i_F/B^i_F$ and the groups $\nu(i)_F$ are

defined. Similarly, we have restriction homomorphisms $r_{L/F}\colon \nu(i)_F \to \nu(i)_L$ and transfer homomorphisms $Tr_{L/F}\colon \nu(i)_L \to \nu(i)_F$ where appropriate.

For any field F of characteristic $p>0$, the group $K_n^M(F)/pK_n^M(F)$ will be denoted by $k_n(F)$. For each integer $n \geq 0$, there is a homomorphism

$$dlog\colon k_n(F) \to \Omega_F^n$$

of abelian groups, called the *differential symbol.* It acts on pure symbols $\{b_1,...,b_n\}$, $b_i \in F^*$, by the formula

$$dlog(\{b_1,...,b_n\}) = \frac{db_1}{b_1} \wedge ... \wedge \frac{db_n}{b_n}.$$

The following result may be viewed as a characteristic p analogue of the p-torsion part of the Bloch-Kato conjecture:

Theorem 2.2 (Bloch-Gabber-Kato, [BK86]). *The sequence*

$$0 \to k_n(F) \xrightarrow{dlog} \Omega_F^n \xrightarrow{\wp} \Omega_F^n/B_F^n$$

is exact for all $n \geq 0$.

In other words, the differential symbol induces an isomorphism $k_n(F) \xrightarrow{\sim} \nu(n)_F$ of abelian groups for all $n \geq 0$. These isomorphisms are evidently compatible with the restriction homomorphisms on the groups $k_n(F)$ and $\nu(n)_F$ induced by an extension of fields $F \subset L$. They are also compatible with transfers:

Proposition 2.3. *Let $F \subset L$ be a finite separable extension of fields of characteristic $p>0$. Then the diagram*

$$\begin{array}{ccc} k_n(L) & \xrightarrow{dlog} & \nu(n)_L \\ {\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle Tr_{L/F}} \\ k_n(F) & \xrightarrow{dlog} & \nu(n)_F \end{array}$$

is commutative for all $n \geq 0$.

Proof. [GS06], Lemma 9.5.4. □

3. Some preliminaries

From now on, unless stated otherwise, *all fields are of fixed characteristic $p>0$.* In this section we will collect some lemmas which will be used to prove the main results. All these results can essentially be found in [Kat82a], though some have been formulated in slightly different terms here.

Let $k \subset F$ be an extension of fields satisfying $F^p \subset L$, and let L be a finite separable extension of k. Put $L' = L \cdot F$. Then L' is a finite separable extension of F. By [Mat89], Theorem 25.1, the natural sequence

$$L' \otimes_L \Omega^1_{L|k} \to \Omega^1_{L'|k} \to \Omega^1_{L'|L} \to 0$$

of L'-vector spaces is exact. But since L is finite separable over k, $\Omega^1_{L|k} = 0$ (cf. [Liu02], §6 Lemma 1.13), and therefore the canonical homomorphism $\Omega^1_{L'|k} \to \Omega^1_{L'|L}$

is an isomorphism. It follows that the canonical homomorphisms $\Omega^i_{L'|k} \to \Omega^i_{L'|L}$ are isomorphisms for all $i \geq 0$. We may therefore define homomorphisms

$$r_{L'|L/F|k} \colon \Omega^i_{F|k} \to \Omega^i_{L'|L} \tag{3.1}$$

and

$$Tr_{L'|L/F|k} \colon \Omega^i_{L'|L} \to \Omega^i_{F|k} \tag{3.2}$$

for all $i \geq 0$ as the compositions

$$\Omega^i_{F|k} \xrightarrow{r_{L'|k/F|k}} \Omega^i_{L'|k} \xrightarrow{\sim} \Omega^i_{L'|L}$$

and

$$\Omega^i_{L'|L} \xrightarrow{\sim} \Omega^i_{L'|k} \xrightarrow{Tr_{L'|k/F|k}} \Omega^i_{F|k}.$$

respectively. Note that the composition $Tr_{L'|L/F|k} \circ r_{L'|L/F|k}$ coincides with $Tr_{L'|k/F|k} \circ r_{L'|k/F|k}$, which is nothing else but multiplication by the degree $[L' : F]$. One may easily check that these maps are functorial and compatible with the differential d and the Artin-Schreier operator $\wp$. By Proposition 2.2, the diagram

$$\begin{CD} k_n(L') @>{dlog}>> \Omega^n_{L'} @>>> \Omega^n_{L'|L} \\ @V{N_{L'/F}}VV @V{Tr_{L'/F}}VV @VV{Tr_{L'|L/F|k}}V \\ k_n(F) @>{dlog}>> \Omega^n_F @>>> \Omega^n_{F|k} \end{CD} \tag{3.3}$$

is commutative.

Lemma 3.1. *Let $k \subset F$ be as above, let L be a finite extension of k of degree prime to p, and put $L' = L \cdot F$. Let $\omega \in \Omega^n_{F|k}$. If $r_{L'|L/F|k}(\omega)$ can be written as a sum of elements of the form $\frac{dc_1}{c_1} \wedge ... \wedge \frac{dc_n}{c_n}$, $c_i \in L'$, then ω can be written as a sum of elements of the form $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_n}{b_n}$, $b_i \in F$.*

Proof. Since $[L : k]$ is prime to p, so is $[L' : F]$. In particular, L' is a separable extension of F. By the commutativity of (3.3), the element $[L' : F]\omega = Tr_{L'|L/F|k} \circ r_{L'|L/F|k}(\omega)$ can be written as a sum of elements of the form $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_n}{b_n}$, $b_i \in F$. Since $[L' : F]$ is invertible modulo p, the same is true of ω. □

The second lemma we will need is:

Lemma 3.2 (cf. [Kat82a], §1, Lemma 3)**.** *Let k be a field, let E be a degree p purely inseparable extension of k, and let $g \colon E \to k$ be a k-linear map. Then, after replacing k by a simple extension of degree prime to p, E by the composite of this extension with itself, and g by the induced map, there exists $c \in E^*$ such that $g(c^i) = 0$ for all $i \in [1, p-1]$.*

For the convenience of the reader, we will include a proof, following [CT99].

Lemma 3.3. *Let $k \subset E$ be as in Lemma 3.2, and let $\omega \in \Omega^1_{E|k} \setminus B^1_{E|k}$. Then, after replacing k by a simple extension of degree $p-1$ (and E by the composite of this extension with itself), there exist $u \in k^*$ and $y \in E^*$ such that $\omega = u\frac{dy}{y}$.*

Proof. Let b be a generator of E over k. Then $\frac{db}{b}$ is a generator of the 1-dimensional E-vector space $\Omega^1_{E|k}$, and so there exists $a \in E^*$ such that $\omega = a\frac{db}{b}$. On the other hand, the k-vector space $\Omega^1_{E|k}/B^1_{E|k}$ is 1-dimensional by Proposition 2.1, and since $\omega \notin B^1_{E|k}$, it is generated by the class of ω. In particular, there exists $\rho \in k^*$ such that

$$a^p\frac{db}{b} = \rho a\frac{db}{b} \pmod{B^1_{E|k}}. \tag{3.4}$$

Let u be a root of $X^{p-1}-\rho$ in a suitable splitting field, and replace k by $k(u)$ and E by $E(u)$. Then, dividing (3.4) by u^p, and putting $c=a/u$, we get

$$(c^p-c)\frac{db}{b} \in B^1_{E|k}.$$

In other words, $c\frac{db}{b} \in \nu(1)_{E|k}$. By a theorem of P. Cartier (cf. [GS06], Theorem 9.3.3), the natural map $E^* \to \nu(1)_{E|k}$ sending y to $dlog(y) = \frac{dy}{y}$ is surjective, and so there exists $y \in E^*$ such that $c\frac{db}{b} = \frac{dy}{y}$. Hence $\omega = a\frac{db}{b} = u\frac{dy}{y}$, as we wanted. □

Lemma 3.4. *Let $k \subset E$ be a finite extension of fields, and let V, W be codimension 1 k-vector subspaces of E. Then there exists $\alpha \in E^*$ such that $\alpha V = W$.*

Proof. We may assume that $E \neq k$. Let $[E:k]=n>1$, and let $\{e_1,...,e_{n-1}$ be a k-basis of E. Let $\alpha \in E^*$. Then $\alpha V = W$ if and only if $\alpha e_i \in W$ for each $i \in [1,n-1]$. Put $W_i = \frac{1}{e_i}W$ for each i. These are again codimension 1 k-vector subspaces of E, and we have $\alpha V = W$ if and only if $\alpha \in W_i$ for all $i \in [1,n-1]$. Therefore, to prove the lemma, it suffices to show that $\dim(\cap_i W_i) > 0$. This is evident. □

Proof of Lemma 3.2. Let b be a generator of E over k. Then $\frac{db}{b}$ is a generator of the 1-dimensional E-vector space $\Omega^1_{E|k}$. Let $\phi\colon \Omega^1_{E|k} \to E$ by the E-linear isomorphism defined by sending the generator $\frac{db}{b}$ to 1, and let $g' = g\circ\phi$. If $g(1)=0$, then the statement holds with $c=1$. Otherwise, $g'(\frac{db}{b}) \neq 0$, and so $\ker(g')$ is a codimension 1 k-vector subspace of $\Omega^1_{E|k}$ for dimension reasons. Now, by Proposition 2.1, $B^1_{E|k}$ is also a codimension 1 k-vector subspace of $\Omega^1_{E|k}$, and so there exists $\alpha \in E^*$ such that $B^1_{E|k} = \alpha\ker(g')$ by Lemma 3.4. It follows that for any $x \in E$, we have $g(x)=0$ if and only if $\alpha x\frac{db}{b} \in B^1_{E|k}$. In particular, since $g(1)\neq 0$, we have $\alpha\frac{db}{b} \notin B^1_{E|k}$. By Lemma 3.3, after replacing k by a simple extension of degree prime to p (and E by the composite of this extension with itself), there exist $u \in k^*$ and $y \in E^*$ such that $\alpha\frac{db}{b} = u\frac{dy}{y}$. Now, for all $i \in [1,p-1]$, i is invertible mod p, and we have

$$\alpha y^i\frac{db}{b} = uy^{i-1}dy = \frac{ud(y^i)}{i} = d(\frac{uy^i}{i}) \in B^1_{E|k}$$

(since $u/i \in k^*$). Hence $g(y^i)=0$ for all $i \in [1,p-1]$, and the lemma is proved. □

Finally, we record the following simple lemma, which will be used several times in section 5:

Lemma 3.5. *Let $k \subset F$ be an extension of fields satisfying $F^p \subset k$, and let $a_1, ..., a_n$ be p-independent elements of F over k. If $a \in F$ satisfies*

$$(a^p - a)\frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} \in B^n_{F|k},$$

then $a \in k(a_1, ..., a_n)$.

Proof. Applying the differential d to the expression

$$(a^p - a)\frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} \in B^n_{F|k},$$

we obtain an equality

$$da \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n}$$

in $\Omega^{n+1}_{F|k}$. This immediately implies that $a \in k(a_1, ..., a_n)$, since otherwise the elements $a, a_1, ..., a_n$ could be extended to a p-basis of F over k. $\square$

4. A special case

Let $k \subsetneq F$ be an extension of fields satisfying $F^p \subset k$. Assume further that F is finite over k, and let $r \geq 1$ be such that $[F : k] = p^r$. Let $\{b_1, ..., b_r\}$ be a p-basis of F over k (in other words, $F = k(b_1, ..., b_r)$). Recall that $\Omega^r_{F|k}$ is a 1-dimensional F-vector space generated by $\frac{db_1}{db} \wedge ... \wedge \frac{db_r}{b_r}$. In what follows, we will usually omit the notation $r_{./.}$ when restricting forms over relative field extensions. The following result is essentially due to K. Kato (cf. [Kat82a]). Since the precise statement we want is not explicitly formulated in [Kat82a], we include a proof for the reader's convenience. We refer to §2.3 for the definition of a logarithmic pure symbol.

Proposition 4.1. *In the above notation, let $\omega \in \nu(r)_{F|k}$. Then ω can be written as a sum of logarithmic pure symbols in $\Omega^r_{F|k}$.*

Proof. We will prove the proposition by induction on r. In view of Lemma 3.1, it will be sufficient to prove the statement up to replacing k by a finite extension of degree prime to p, and L by the composite of this extension with itself. Write η for the generator $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_r}{b_r}$ of the F-vector space $\Omega^r_{F|k}$, and let $a \in F$ be such that $\omega = a\eta$. Let $k_1 = k(b_1)$. Consider the k-linear map $k_1 \to \Omega^r_{F|k}/B^r_{F|k}$ defined by sending $x \in k_1$ to the element $x\omega$. Now, since $\Omega^r_{K|k}/B^r_{K|k}$ is a 1-dimensional k-vector space, Lemma 3.2 implies that there is an extension $k \subset L$ of degree prime to p, such that if $L_1 = L \cdot k_1$ and $L' = L \cdot F$ we have

$$c^i\omega \in B^r_{L'|L} \quad (\forall i \in [1, p-1]) \tag{4.1}$$

for some $c \in L_1^*$ (we are writing ω in place of $r_{L'|L/F|k}(\omega)$ here) Note that $\{b_1, ..., b_r\}$ is still a p-basis of L' over L. We claim that $c \notin L$. Indeed, if $c \in L$, then (4.1) (in the case $i = 1$) implies that $\omega \in B^r_{L'|L}$. Since $\omega \in \nu(r)_{F|k}$, we have $(a^p - a)\eta \in B^r_{F|k}$, and hence

$$\begin{aligned} 0 \equiv (a^p - a)\eta &\equiv a^p\eta - \omega \\ &\equiv a^p\eta \pmod{B^r_{L'|L}}, \end{aligned}$$

and so $\eta \in B^r_{L'|L}$, which is impossible, because η is a generator of the 1-dimensional L-vector space $\Omega^r_{L'|L}/B^r_{L'|L}$ (cf. Proposition 2.1). It follows that $L_1 = L(c)$, and so $\{c, b_2, ..., b_r\}$ is a p-basis of L' over L. Let η' denote the element $\frac{db_2}{b_2} \wedge ... \wedge \frac{db_r}{b_r}$, and let $a' \in L'$ be such that

$$\omega = \begin{cases} a'\frac{dc}{c} & \text{if } r = 1 \\ a'\frac{dc}{c} \wedge \eta' & \text{if } r > 1 \end{cases} \tag{4.2}$$

as an element of $\Omega^r_{L'|L}$. Consider the decomposition of L into a direct sum $L_1 \oplus V$ of L-vector subspaces, where V is generated by the products $b_2^{\alpha_2}...b_r^{\alpha_r}$ for all nonzero multiindices $(\alpha_2, ..., \alpha_r) \in [0, p-1]^{r-1}$. Since $L_1 = L(c)$, we can write $a' = \sum_{j=0}^{p-1} \lambda_j c^j + v$ for some $v \in V$ and $\lambda_i \in L$. We claim that $\lambda_j = 0$ for all $j \in [1, p-1]$. To see this, suppose that $\lambda_j \neq 0$ for some $j \in [1, p-1]$. Consider the direct sum decomposition

$$\Omega^r_{L'|L} = \oplus_\alpha \Omega^r_{L'|L}(\alpha)$$

of (2.2) with respect to the ordered p-basis $\{c, b_2, ..., b_r\}$ of L' over L. By (4.2), the projection of the element $c^{p-j}\omega \in \Omega^r_{L'|L}$ to the component $\Omega^r_{L'|L}(0)$ is equal to $\lambda_j \frac{dc}{c} \wedge \eta'$ (or $\lambda_j \frac{dc}{c}$ if $r = 1$), which is nonzero by assumption. But $c^{p-j}\omega \in B^r_{L'|L}$ by (4.1), and this contradicts the fact that the differentials of the complex $\Omega^\bullet_{L'|L}(0)$ are trivial. We conclude that $\lambda_j = 0$ for all $j \in [1, p-1]$, and so we have

$$a' = \lambda_0 + v \tag{4.3}$$

with $\lambda_0 \in L$ and $v \in V$. Now, since $\wp(\omega) = 0$, (4.2) implies that

$$(a'^p - a)\frac{dc}{c} \wedge \eta' \in B^r_{L'|L}$$

(or simply $(a'^p - a)\frac{dc}{c} \in B^r_{L'|L}$ if $r = 1$). Now, by (4.3), the projection of $(a'^p - a')\frac{dc}{c} \wedge \eta' \in \Omega^r_{L'|L}$ (resp. $(a'^p - a')\frac{dc}{c} \in \Omega^r_{L'|L}$ if $r = 1$) to the component $\Omega^r_{L'|L}(0)$ is equal to

$$(a'^p - \lambda_0)\frac{dc}{c} \wedge \eta'$$

(resp. $(a'^p - \lambda_0)\frac{dc}{c}$ if $r = 1$). But the subcomplex $\Omega^\bullet_{L'|L}(0)$ has trivial differentials, and so we must have $a'^p = \lambda_0$. Hence

$$a'^p - a' = -v. \tag{4.4}$$

In the case where $r = 1$, $V = 0$, and so $a'^p = a'$. In other words, $a' \in \mathbb{Z}/p\mathbb{Z}$, and hence we have

$$\omega = a'\frac{dc}{c} = \frac{d(c^{a'})}{c^{a'}}.$$

This proves the case $r = 1$. If $r > 1$, consider now the direct sum decomposition

$$\Omega^r_{L'|L_1} = \oplus_\alpha \Omega^r_{L'|L_1}(\alpha)$$

of (2.2) with respect to the ordered p-basis $\{b_2, ..., b_r\}$ of L' over L_1. Since $[L' : L_1] = p^{r-1}$, Proposition 2.1 implies that $v\eta' \in B^{r-1}_{L'|L_1}$. Hence

$$(a'^p - a')\eta' \in B^{r-1}_{L'|L_1}$$

by (4.4). In other words, $a'\eta' \in \nu(r-1)_{L'|L_1}$. By the induction hypothesis, $a'\eta'$ can be written as a sum of logarithmic pure symbols in $\Omega_{L'|L_1}$. Now, from the standard exact sequence ([Mat89], Theorem 25.1)

$$L' \otimes_{L_1} \Omega^1_{L_1|L} \to \Omega^1_{L'|L} \to \Omega^1_{L'|L_1} \to 0$$

we see that the kernel of the canonical surjection $\Omega^{r-1}_{L'|L} \to \Omega^{r-1}_{L'|L_1}$ is equal to $\frac{dc}{c} \wedge \Omega^{r-2}_{L'|L}$. In particular, it is annihilated by $\frac{dc}{c}$. It follows that $\omega = \frac{dc}{c} \wedge (a'\eta')$ can be written as a sum of logarithmic pure symbols in $\Omega^r_{L'|L}$. This completes the proof. □

Remark 4.2. Proposition 4.1 will serve as the basis for our proof of Theorem 1.4. In fact, it is not difficult to see that Proposition 4.1 is already sufficient to prove Theorem 1.4 in the special case where $m = n$.

5. The general case

In this section we prove Theorem 1.4. The proof employs adaptions of the methods used by K. Kato in [Kat82a] to prove the surjectivity of the differential symbol. The general idea is very similar to that of Proposition 4.1, but we are now dealing with more complicated elements in spaces of differential forms. Let $k \subsetneq F$ be an extension of fields satisfying $F^p \subset k$. Suppose that we have p-independent elements $a_1, ..., a_n$ of F over k for some positive integer n. Suppose further that F is finite over k, and let $r \geq 0$ be such that $[F : k] = p^{n+r}$. We may extend $a_1, ..., a_n$ to a p-basis $\{b_1, ..., b_r, a_1, ..., a_n\}$ of F over k. If $r \geq 1$ and $j \in [1, r]$, we will define S_j to be the set of all strictly increasing functions $[1, j] \to [1, r]$. In this case, for each $s \in S_j$, we define a logarithmic pure symbol

$$\omega_s = \frac{db_{s(1)}}{b_{s(1)}} \wedge ... \wedge \frac{db_{s(j)}}{b_{s(j)}} \in \Omega^j_{F|k}.$$

We endow the sets S_j with the lexicographical ordering, and given $s \in S_j$, we define $\Omega^j_{F|k,<s}$ to be the F-vector subspace of $\Omega^j_{F|k}$ generated by those symbols ω_t, $t \in S_j$, for which $t < s$ (if there are no such t, we set $\Omega^j_{F|k,<s} = 0$). We will further define $S_0 = \{s_0\}$ to be a one element set with $\omega_{s_0} = 1$ and $\Omega^0_{F|k,<s_0} = 0$. Now, for any $j \geq 1$ and $s \in S_{j-1}$, we can define $B^j_{F|k,<s} = d(\Omega^{j-1}_{F|k,<s}) \subset \Omega^j_{F|k}$.

Note that if $k \subset L$ is a finite extension of degree prime to p, and $L' = L \cdot F$, the set $\{b_1, ..., b_r, a_1, ..., a_n\}$ is again a p-basis for L' over L. As in section 4, we will often omit the notation $r_{./.}$ when restricting differential forms over relative extensions.

Proposition 5.1. *In the above notation, let $m \in [n, r+n]$, and suppose that we have $s \in S_{m-n}$ and $a \in F$ satisfying*

$$(a^p - a)\omega_s \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} \in \mathrm{span}_F(\Omega^{m-n}_{F|k,<s'} \wedge \Omega^n_{F|k}) + B^m_{F|k}. \tag{5.1}$$

Then there is a finite extension $k \subset L$ of degree prime to p, such that if $L' = L \cdot F$, there exists $\nu \in \Omega^{m-n}_{L'|L,<s}$ such that the element

$$a\omega_s \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} - \nu \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n}$$

can be written as a sum of logarithmic pure symbols β in $\Omega^m_{L'|L}$. Moreover, each such symbol β can be chosen to satisfy $\beta = \lambda\omega_s \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n}$ for some $\lambda \in L$.

Proof. We write η for the element $\frac{a_1}{a_1} \wedge ... \wedge \frac{a_n}{a_n} \in \Omega^n_{F|k}$. If $m > n$, define subfields of F as follows:

$$\begin{aligned} k_0 &= k(b_1, ..., b_{s(1)-1}),\\ k_1 &= k_0(b_{s(1)}),\\ k_2 &= k_1(b_{s(1)+1}, b_{s(2)+2}, ..., b_{s(m-n)}, a_1, ..., a_n) \end{aligned}$$

(where $k_0 = k$ if $s(1) = 1$). If $m = n$, we set $k_0 = k_1 = k$ and $k_2 = k(a_1, ..., a_n)$. Let N be such that $[k_2 : k_0] = p^N$, and let $l(1) < ... < l(N-m)$ be the sequence of integers in the interval $[s(1), s(m-n)]$ which are not in the image of s. Define

$$\omega_c = \frac{db_{l(1)}}{b_{l(1)}} \wedge ... \wedge \frac{db_{l(N-m)}}{b_{l(N-m)}} \in \Omega^{N-m}_{F|k}$$

and set $\omega_{max} = \omega_c \wedge \omega_s \wedge \eta \in \Omega^N_{F|k}$ (if $N = m$, then we set $\omega_c = 1 \in \Omega^0_{F|k}$ and $\omega_{max} = \omega_s \wedge \eta$). Taking the wedge product of (5.1) with ω_c, we get

$$(5.2) \qquad (a^p - a)\omega_{max} \in \omega_c \wedge \mathrm{span}_F(\Omega^{m-n}_{F|k,<s} \wedge \Omega^n_{F|k}) + \omega_c \wedge B^m_{F|k}.$$

By the definitions of k_0 and ω_c, the image of $\omega_c \wedge \Omega^{m-n}_{F|k,<s} \wedge \Omega^n_{F|k}$ under the canonical projection

$$\Omega^N_{F|k} \to \Omega^N_{F|k_0}$$

is trivial. Since $\omega_c \wedge B^m_{F|k} \subset B^N_{F|k}$, it therefore follows from (5.2) that after projecting to $\Omega^N_{F|k_0}$, we have $(a^p - a)\omega_{max} \in B^N_{F|k_0}$. It follows from Lemma 3.5 that $a \in L_2$. Now, using a direct sum decomposition of the form (2.2), we can rewrite (5.1) as

$$(a^p - a)\omega_s \wedge \eta \in \mathrm{span}_{k_2}(\Omega^{m-n}_{k_2|k_0,<s} \wedge \Omega^n_{k_2|k_0}) + B^m_{k_2|k_0}$$

(note that we are now working relative to k_0). We can therefore find $\omega_1 \in \Omega^{m-1}_{k_2|k_0}$, and elements $\omega_2^l \in \Omega^{m-n}_{k_2|k_0,<s}$, $\omega_3^l \in \Omega^n_{k_2|k_0}$ such that

$$(5.3) \qquad (a^p - a)\omega_s \wedge \eta = \sum_l (\omega_2^l \wedge \omega_3^l) + d(\omega_1) \in \Omega^m_{k_2|k_0}.$$

We may further assume that the elements ω_3^l are logarithmic pure symbols. In particular, $d(\omega_3^l) = 0$ for all l. The proof now proceeds by induction on $m - n$. If $m - n = 0$, then $k_2 = k(a_1, ..., a_n)$, and (5.3) becomes

$$(a^p - a)\eta \in B^n_{k_2|k}.$$

In other words, $a\eta \in \nu(n)_{k_2|k}$. By Proposition 4.1, $a\eta$ can be written as a sum of logarithmic pure symbols $\frac{de_1}{e_1} \wedge ... \wedge \frac{de_n}{e_n}$ in $\Omega^n_{k_2|k_0}$. Since $[k_2 : k] = p^n$, we must have $k(e_1, ..., e_n) = k(a_1, ..., a_n)$ for each such nonzero symbol. In particular, there exists $\lambda \in k_2$ such that $\frac{de_1}{e_1} \wedge ... \wedge \frac{de_n}{e_n} = \lambda\eta$ in $\Omega^n_{k_2|k_0}$. The same is clearly true after restricting to $\Omega^n_{F|k}$, so this proves the base $m - n = 0$ of the induction.

Assume now that $m-n>0$. Since $\omega_c \wedge \Omega^{m-n}_{k_2|k_0,<s} \wedge \Omega^n_{k_2|k_0} = 0$, taking the wedge product of (5.3) with ω_c gives

$$(a^p - a)\omega_{max} = \omega_c \wedge d(\omega_1) \in B^N_{k_2|k_0}. \tag{5.4}$$

Now, consider the k-linear map $k_1 \to \Omega^N_{k_2|k_0}/B^N_{k_2|k_0}$ defined by sending $x \in k_1$ to the class of $xa\omega_{max}$. Since $[k_2 : k_0] = p^N$, $\Omega^N_{k_2|k_0}/B^N_{k_2|k_0}$ is a 1-dimensional k_0-vector space generated by the class of ω_{max} by Proposition 2.1. By Lemma 3.2, there is a simple extension $k_0 \subset L_0$ of degree prime to p, such that if $L_1 = L_0 \cdot k_1$, $L_2 = L_0 \cdot k_2$, we have

$$c^i a\omega_{max} \in B^N_{L_2|L_0} \quad (\forall i \in [1, p-1])$$

for some $c \in L_1^*$. Let u be a generator of L_0 over k_0, and put $L = k(u)$ and $L' = F(u)$. Note that $k \subset L$ is also an extension of degree prime to p, since $k \subset k_0$ is purely inseparable. Now, we claim that $c \notin L_0$. Indeed, if $c \in L_0$, then since $ca\omega_{max} \in B^N_{L_2|L_0}$, we have $a\omega_{max} \in B^N_{L_2|L_0}$. Equation (5.4) then implies that $a^p\omega_{max} \in B^N_{L_2|L_0}$, from which it follows that $\omega_{max} \in B^N_{L_2|L_0}$. This is not possible, since ω_{max} is a generator of the 1-dimensional L_0-vector space $\Omega^N_{L_2|L_0}/B^N_{L_2|L_0}$. We conclude that $L_1 = L_0(c)$, and so $\{c, b_{s(1)+1}, b_{s(1)+2}, ..., b_{s(m-n)}, a_1, ..., a_n\}$ is a p-basis of L_2 over L_0. If $m-n>1$, write S'_{m-n} for the set of all increasing functions $[2, m-n] \to [1, r]$, and endow S'_{m-n} with the lexicographical ordering We let $s' \in S'_{m-n}$ denote the restriction of s to $[2, m-n]$, and write $\omega_{s'}$ for the element $\frac{db_{s(2)}}{b_{s(2)}} \wedge \frac{db_{s(3)}}{b_{s(3)}} \wedge ... \wedge \frac{db_{s(m-n)}}{b_{s(m-n)}} \in \Omega^{m-n-1}_{L'|L}$. If $m-n=1$, we put $\omega_{s'} = 1 \in \Omega^0_{L'|L}$. Let $a' \in L_2$ be such that

$$a\omega_s \wedge \eta = a'\frac{dc}{c} \wedge \omega_{s'} \wedge \eta \tag{5.5}$$

as an element of $\Omega^m_{L_2|L_0}$. We are going to show that

$$(a'^p - a')\omega_{s'} \wedge \eta \in \mathrm{span}_{L'}(\Omega^{m-n-1}_{L'|L,<s'} \wedge \Omega^n_{L'|L}) + B^{m-1}_{L'|L}. \tag{5.6}$$

Write L_2 as a direct sum $L_1 \oplus V$ of L_0-vector subspaces, where V is generated by the products $b^{\alpha_2}_{s(1)+1} b^{\alpha_3}_{s(1)+2} ... b^{\alpha_{N-n}}_{s(m-n)} a_1^{\alpha_{N-n+1}} ... a_n^{\alpha_N}$ for all nontrivial multiindices $\alpha = (\alpha_2, ..., \alpha_N) \in [0, p-1]^{N-1}$. Since $L_1 = L_0(c)$, we can write $a' = \sum_{j=1}^{p-1} \lambda_j c^j + v$, where $\lambda_j \in L_0$ and $v \in V$. We claim that $\lambda_j = 0$ for all $j \in [1, p-1]$. To see this, suppose that $\lambda_j \neq 0$ for some $j \in [1, p-1]$. Consider the direct sum decomposition

$$\Omega^N_{L_2|L_0} = \oplus_\alpha \Omega^N_{L_2|L_0}(\alpha)$$

of (2.2) with respect to the ordered p-basis $\{c, b_{s(1)+1}, ..., b_{s(m-n)}, a_1, ..., a_n\}$ of L_2 over L_0. The projection of the element $c^{p-j}a\omega_{max}$ to the component $\Omega^N_{L_2|L_0}(0)$ is equal to $\lambda_j a\omega_{max}$, which is nonzero by assumption. On the other hand, we have $c^{p-j}a\omega_{max} \in B^N_{L_2|L_0}$, and this is impossible because the complex $\Omega^\bullet_{L_2|L_0}(0)$ has trivial differentials. Hence $\lambda_j = 0$ for all $j \in [1, p-1]$, and we have

$$a' = \lambda_0 + v \tag{5.7}$$

with $\lambda_0 \in L_0$ and $v \in V$. Applying the Artin-Schreier operator $\wp\colon \Omega^m_{L_2|L_0} \to \Omega^m_{L_2|L_0}/B^m_{L_2|L_0}$ to both sides of (5.5), we have

$$(a^p - a)\omega_s \wedge \eta = (a'^p - a')\frac{dc}{c} \wedge \omega_{s'} \wedge \eta \pmod{B^m_{L_2|L_0}}.$$

Restricting (5.3) to $\Omega^m_{L_2|L_0}$, we therefore obtain

$$(a'^p - a')\frac{dc}{c} \wedge \omega_{s'} \wedge \eta \in \sum_l (\omega_2^l \wedge \omega_3^l) + B^m_{L_2|L_0}. \tag{5.8}$$

Since $\omega_c \wedge \Omega_{L_2|L_0,<s} \wedge \Omega^n_{L_2|L_0} = 0$, taking the wedge product of both sides of (5.8) with ω_c gives

$$(a'^p - a')\frac{dc}{c} \wedge \omega_{s'} \wedge \eta \wedge \omega_c \in B^N_{L_2|L_0}. \tag{5.9}$$

Now, the projection of the element $(a'^p - a)\frac{dc}{c} \wedge \omega_{s'} \wedge \eta \wedge \omega_c$ to the component $\Omega^N_{L_2|L_0}(0)$ is equal to

$$(a'^p - \lambda_0)\frac{dc}{c} \wedge \omega_{s'} \wedge \eta \wedge \omega_c.$$

Since the complex $\Omega^\bullet_{L_2|L_0}(0)$ has trivial differentials, (5.9) implies that $a'^p - \lambda_0 = 0$. Hence (5.7) becomes

$$a'^p - a' = -v. \tag{5.10}$$

Applying the differential $d\colon \Omega^m_{L_2|L_0} \to \Omega^{m+1}_{L_2|L_0}$ to (5.8), and using the fact that $d(\omega_3^l) = 0$ for all l, we get

$$d(v\omega_s') \wedge \frac{dc}{c} \wedge \eta = \sum_l d(-\omega_2^l \wedge \omega_3^l) = \sum_l d(-\omega_2^l) \wedge \omega_3^l$$

(we also used (5.10) here). Since $\frac{dc}{c}$ differs from $\frac{db_{s(1)}}{b_{s(1)}}$ by a constant in L_1^*, it is easy to see that $\Omega^{m-n}_{L_2|L_0,<s} \subset \Omega^{m-n-1}_{L_2|L_0,<s'} \wedge \frac{dc}{c}$. Hence we can find elements $\omega_4^l \in \Omega^{m-n-1}_{L_2|L_0,<s'}$ such that

$$d(v\omega_{s'}) \wedge \frac{dc}{c} \wedge \eta = \sum_l d(\omega_4^l) \wedge \frac{dc}{c} \wedge \omega_3^l \in \Omega^{m+1}_{L_2|L_0}.$$

Rearranging, we obtain the following equality in $\Omega^{m+1}_{L_2|L_0}$:

$$d\left(v\omega_{s'} \wedge \eta - \sum_l (\omega_4^l \wedge \omega_3^l)\right) \wedge \frac{dc}{c} = 0.$$

Since c is part of a p-basis for L_2 over L_0, it follows that the element

$$\omega_5 = d\left(v\omega_{s'} \wedge \eta - \sum_l (\omega_4^l \wedge \omega_3^l)\right) \in \Omega^m_{L_2|L_0}$$

is divisible by $\frac{dc}{c}$. Since $c \in L_1$, the image of ω_5 under the canonical projection $\Omega^m_{L_2|L_0} \to \Omega^m_{L_2|L_1}$ is trivial. Now, consider the direct sum decomposition

$$\Omega^{m-1}_{L_2|L_1} = \oplus_\alpha \Omega^{m-1}_{L_2|L_1}(\alpha)$$

of (2.2) with respect to the ordered p-basis $\{b_{s(1)+1}, b_{s(1)+2}, ..., b_{s(m-n)}, a_1, ..., a_n\}$ of L_2 over L_1. By the definition of V, the projection of the element $\nu\omega_{s'} \wedge \eta$ to the component $\Omega^{m-1}_{L_2|L_1}(0)$ is zero. This need not be true for the elements $\omega_4^l \wedge \omega_3^l$, but observe that since the differentials of $\Omega^\bullet_{L_2|L_1}(0)$ are trivial, we can modify the ω_4^l by suitable elements of $\Omega^{m-n-1}_{L_2|L_1}$ without affecting the fact that ω_5 projects trivially to $\Omega^m_{L_2|L_1}$. We can therefore assume that the image of $v\omega_{s'} \wedge \eta - \sum_l(\omega_4^l \wedge \omega_3^l)$ under the projection to the component $\Omega^{m-1}_{L_2|L_1}(0)$ is zero. By Proposition 2.1, it follows that we have

$$v\omega_{s'} \wedge \eta - \sum_l (\omega_4^l \wedge \omega_3^l) \in B^{m-1}_{L_2|L_1}.$$

Therefore

$$v\omega_{s'} \wedge \eta \in \mathrm{span}_{L_2}(\Omega^{m-n-1}_{L_2|L_1,<s'} \wedge \Omega^n_{L_2|L_1}) + B^{m-1}_{L_2|L_1}.$$

Passing back to $\Omega^{m-1}_{L_2|L}$, we have

$$v\omega_{s'} \wedge \eta \in \mathrm{span}_{L_2}(\Omega^{m-n-1}_{L_2|L,<s'} \wedge \Omega^n_{L_2|L}) + B^{m-1}_{L_2|L} + \ker(\Omega^{m-1}_{L_2|L} \to \Omega^{m-1}_{L_2|L_1}).$$

But it is not difficult to see, from the definition of L_1, that $\ker(\Omega^{m-1}_{L_2|L} \to \Omega^{m-1}_{L_2|L_1}) \subset \mathrm{span}_{L_2}(\Omega^{m-n-1}_{L_2|L,<s'} \wedge \Omega^n_{L_2|L})$. Hence

$$v\omega_{s'} \wedge \eta \in \mathrm{span}_{L_2}(\Omega^{m-n-1}_{L_2|L,<s'} \wedge \Omega^n_{L_2|L}) + B^{m-1}_{L_2|L}.$$

Restricting to $\Omega_{L'|L}$, we see that (5.6) holds. By the induction hypothesis, after replacing L by an extension of degree prime to p (and L_0, L_1, L_2, L' by the composite of this extension with themselves respectively) there exists $\nu' \in \Omega^{m-n-1}_{L'|L,<s'}$ such that

$$a'\omega_{s'} \wedge \eta - \nu' \wedge \eta \tag{5.11}$$

can be written as a sum of logarithmic pure symbols β' in $\Omega^{m-1}_{L'|L}$. Moreover, the symbols β' can be chosen so that $\beta' = \lambda'\omega_{s'} \wedge \eta$ in $\Omega^{m-1}_{L'|L}$ for some $\lambda' \in L'$. Since $\frac{dc}{c}$ and $\frac{db_{s(1)}}{b_{s(1)}}$ differ by a constant in L', we have $\frac{dc}{c} \wedge \nu' \in \Omega^{m-n}_{L'|L,<s}$. Now, by (5.5), the equality

$$a\omega_s \wedge \eta = a'\frac{dc}{c} \wedge \omega_{s'} \wedge \eta$$

holds in $\Omega^m_{L'|L_0}$. Hence there is $\omega_7 \in \ker(\Omega^m_{L'|L} \to \Omega^m_{L'|L_0})$ such that the equality

$$a\omega_{s'} \wedge \eta = a'\frac{dc}{c} \wedge \omega_{s'} \wedge \eta + \omega_7 \tag{5.12}$$

holds in $\Omega^m_{L'|L}$. By the definition of L_0, we have $\omega_7 \in \mathrm{span}_{L'}(\Omega^{m-n}_{L'|L,<s} \wedge \Omega^n_{L'|L})$. On the other hand, since $a_1, ..., a_n$ is part of a p-basis of L' over L, equation (5.12) then implies that we can find $\omega_8 \in \Omega^{m-n}_{L'|L,<s}$ such that $\omega_7 = \omega_8 \wedge \eta$ in $\Omega^m_{L'|L}$. Put $\nu = \frac{dc}{c} \wedge \nu' - \omega_8 \in \Omega^{m-n}_{L'|L,<s}$. Then, taking the wedge product of (5.11) with $\frac{dc}{c}$, we see that

$$a\omega_s \wedge \eta - \nu \wedge \eta$$

can be written as a sum of logarithmic pure symbols β in $\Omega^m_{L'|L}$. Moreover, since $\frac{dc}{c}$ and $\frac{db_{s(1)}}{b_{s(1)}}$ differ by a constant in L', the symbols β can be chosen so that $\beta = \lambda\omega_s \wedge \eta$ for some $\lambda \in L'$. The induction step is complete, and the proposition is proved. $\square$

Corollary 5.2. *In the notation preceding Proposition 5.1, let $m \in [n, r+n]$. Suppose we have $\omega \in \Omega^{m-n}_{F|k}$ such that*

$$\omega \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} \in \nu(m)_{F|k}. \tag{5.13}$$

Then there is a finite extension $k \subset L$ of degree prime to p, such that if $L' = L \cdot F$, the element

$$\omega \wedge \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n}$$

can be written as a sum of logarithmic pure symbols β in $\Omega^m_{L'|L}$. Moreover, each such β can be chosen to satisfy $\beta = \lambda \frac{df_1}{f_1} \wedge ... \wedge \frac{df_{m-n}}{f_{m-n}} \wedge \frac{da_1}{a_1} \wedge \frac{da_n}{a_n}$ for some elements $\lambda \in L'$ and $f_1, ..., f_{m-n} \in F$.

Proof. Again, we write η for the element $\frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n} \in \Omega^n_{F|k}$. Let $s \in S_{m-n}$ be minimal such that there exists $a \in F$ and $\nu' \in \Omega^{n-m}_{F|k,<s}$ for which

$$\omega \wedge \eta - a\omega_s \wedge \eta - \nu' \wedge \eta \tag{5.14}$$

can be written as a sum of logarithmic pure symbols β with the special property given the statement of the corollary (clearly such s exists). The Artin-Schreier operator $\wp\colon \Omega^m_{F|k} \to \Omega^m_{F|k}/B^m_{F|k}$ clearly maps $\Omega^{m-n}_{F|k,<s} \wedge \eta$ into its own image modulo $B^m_{F|k}$, and since $\omega \wedge \eta \in \nu(m)_{F|k}$, applying $\wp$ to (5.14) shows that

$$(a^p - a)\omega_s \wedge \eta \in (\Omega^{m-n}_{F|k,<s} \wedge \eta) + B^m_{F|k}$$

(we are using the fact that $\wp$ annihilates logarithmic pure symbols). By Proposition 5.1, there is a finite extension $k \subset L$ of degree prime to p, such that if $L' = L \cdot F$, we can find $\nu \in \Omega^{m-n}_{L'|L,<s}$ such that

$$a\omega_s \wedge \eta - \nu \wedge \eta \tag{5.15}$$

can be written as a sum of logarithmic pure symbols β satisfying the special property in the statement of the corollary. Put $\nu'' = \nu + \nu' \in \Omega^{m-n}_{L'|L,<s}$. By (5.14), we see that

$$\omega \wedge \eta - \nu'' \wedge \eta$$

can be written as a sum of logarithmic pure symbols β satisfying the property in the statement. In particular, we can find $t < s$ such that

$$\omega \wedge \eta - a\omega_t \wedge \eta - \nu' \wedge \eta$$

can be written as a sum of logarithmic pure symbols β of the prescribed type. Thus after making our prime to p extension of $k \subset F$, s is no longer minimal with respect to (5.14). After repeating this procedure finitely many times, we eventually arrive at the situation where $\omega \wedge \eta$ can be written as a sum of logarithmic pure symbols β with the special property in the statement of the corollary. □

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. Again, we set $\eta = \frac{da_1}{a_1} \wedge ... \wedge \frac{da_n}{a_n}$. Assume first that $b_1, ..., b_m \in F^*$ are such that $F^p(a_1, ..., a_n) \subset F^p(b_1, ..., b_m)$. If $[F^p(b_1, ..., b_m) : F^p] < p^m$, then $dlog(\{b_1, ..., b_n\}) = 0$, and so trivially belongs to $\Omega^{m-n}_F \wedge \eta$. Otherwise, we can extend $a_1, ..., a_n$ to a p-basis $\{a_1, ..., a_n, c_1, ..., c_{m-n}\}$ of $F^p(b_1, ..., b_m)$ over F^p. Since

$\{b_1, ..., b_m\}$ is also a p-basis for this extension, the elements $\frac{dc_1}{c_1} \wedge ... \wedge \frac{dc_{m-n}}{c_{m-n}} \wedge \eta$ and $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_m}{b_m}$ are proportional in Ω^m_F. In particular, $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_m}{b_m} \in \Omega^{m-n}_F \wedge \eta$.

Conversely, let $\omega \in \nu(m)_F \cap (\Omega^{m-n}_F \wedge \eta)$. Let $k = F^p$. In order to prove the theorem, we may clearly replace $k \subset F$ by a finite subextension containing $a_1, ..., a_n$. (note here that we are replacing F, but not k; in particular, we have $F^p \subset k$, but these fields need not be equal) Let $[F : k] = p^{r+n}$, and let $c_1, ..., c_r$ be such that $\{c_1, ..., c_r, a_1, ..., a_n\}$ is a p-basis of F over k. If $m > r + n$, then $\omega = 0$, and there is nothing to prove. We can therefore assume that $m \in [n, r + n]$. Since $\omega \in \nu(m)_{F|k}$, Corollary 5.2 implies that, there is a finite extension $k \subset L$ of degree prime to p, such that if $L' = L \cdot F$, we can write

$$\omega = \sum_l \beta^l \tag{5.16}$$

for some logarithmic pure symbols $\beta^l \in \Omega^m_{L'|L}$ satisfying the following property: there exist $\lambda^l \in L'$ and $f^l_1, ..., f^l_{m-n} \in F$ such that $\beta^l = \lambda^l \frac{df^l_1}{f^l_1} \wedge ... \wedge \frac{df^l_{m-n}}{f^l_{m-n}} \wedge \eta$ in $\Omega^m_{L'|L}$. Applying the transfer $Tr_{L'|L/F|k}$ (cf. §3) to both sides of (5.16), we get

$$[L' : L]\omega = \sum_l Tr_{L'|L/F|k}(\beta^l)$$

in $\Omega^m_{F|k}$. Since $[L' : L]$ is invertible modulo p, we have

$$\omega = [L' : L]^{-1} \sum_l Tr_{L'|L/F|k}(\beta^l). \tag{5.17}$$

By the commutativity of (3.3) and Theorem 2.2, we have $Tr_{L'|L/F|k}(\beta^l) \in \nu(m)_{F|k}$ for all l. On the other hand, since the f^l_i and a_i belong to F, we have $Tr_{L'|L/F|k}(\beta^l) = Tr(\lambda^l)\frac{df^l_1}{f^l_1} \wedge ... \wedge \frac{df^l_{m-n}}{f^l_{m-n}} \wedge \eta$ by the definition of the transfer. Thus, in particular, we have $Tr_{L'|L/F|k}(\beta^l) \in \nu(m)_{F|k} \cap (\Omega^{m-n}_{F|k} \wedge \eta)$. In view of (5.17), it will therefore be enough to treat the case where

$$\omega = \lambda\frac{df_1}{f_1} \wedge ... \wedge \frac{df_{m-n}}{f_{m-n}} \wedge \eta. \tag{5.18}$$

for some $\lambda, f_i \in F$. Since $\omega \in \nu(m)_{F|k}$, we have $\lambda \in k(f_1, ..., f_{m-n}, a_1, ..., a_n)$ by Lemma 3.5. We may now replace F by $k(f_1, ..., f_{m-n}, a_1, ..., a_n)$ (this does not change the fact that $\omega \in \nu(m)_{F|k}$). By (5.18), we have

$$(\lambda^p - \lambda)\frac{df_1}{f_1} \wedge ... \wedge \frac{df_{m-n}}{f_{m-n}} \wedge \eta \in B^m_{F|k}.$$

By Proposition 4.1, ω may be written as a sum of (nonzero) logarithmic pure symbols $\frac{db_1}{b_1} \wedge ... \wedge \frac{db_m}{b_m}$ in $\Omega^m_{F|k}$. Since $[F : k] = p^m$, and these symbols are nonzero, we must have $k(b_1, ..., b_m) = F$. In particular, $k(a_1, ..., a_n) \subset k(b_1, ..., b_m)$. This completes the proof of Theorem 1.4. □

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School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom

E-mail address: pmxss4@nottingham.ac.uk